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Laughlin states on the Poincaré half-plane and their quantum group symmetry

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Abstract. We find the Laughlin states of the electrons on the Poincaré half-plane in different representations. In each case we show that a quantum group $su_q(2)$ symmetry exists such that the Laughlin states are a representation of it. We calculate the corresponding filling factor by using the plasma analogy of the fractional quantum Hall effect.

1. Introduction

Studying the behaviour of charged particles on the two-dimensional surface in the presence of a strong magnetic field has led to the discovery of the fractional quantum Hall effect (FQHE) [1, 2]. To explain this phenomenon, Laughlin proposed a suitable N-particle wavefunction which describes the FQHE of the filling factor $\nu = 1/m$, where m is an odd integer [3]. Laughlin's model also has a beautiful analogy with an incompressible fluid of interacting plasma.

Later, the quantum mechanics of the non-relativistic particles in a uniform magnetic field were studied for different two-dimensional surfaces. The first was the sphere on which the magnetic field was produced by a magnetic monopole [4], and recently the topological torus [5] and arbitrary two-dimensional compact Riemann surfaces were studied [6].

One of the important point in the physics of the FQHE is to understand the incompressibility feature of this problem in the language of the symmetries of this theory. In [7], it is shown that this feature relates to the existence of the Fairlie–Fletcher–Zachos (FFZ) algebra [8] as a symmetry algebra of the Hamiltonian. As this algebra reduces to the area-preserving diffeomorphism it can explain the incompressibility. It was also shown that the generators of the FFZ algebra, which are the magnetic translation operators, could represent the $su_q(2)$ algebra where q is a function of the magnetic field [5, 7, 9].

The case of non-compact surfaces, and especially the upper half-plane with Poincaré metric was also studied in several papers [10–12]. In these articles the one-particle wavefunctions and the symmetries of the Hamiltonian were discussed. In [13] we began our investigation of $su_q(2)$ symmetry for this surface by finding the generators of this quantum algebra and showing that the one-particle ground state is a representation of this $su_q(2)$.

In this paper we are going to complete our study of the FQHE on the Poincaré half-plane by calculating the Laughlin states. We will find different representations of this state. To

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clarify what we mean by this we remind the reader that in the original work of Laughlin the ground states were the eigenstates of the angular momentum. But in our case the angular momentum is not the symmetry of the Hamiltonian, nevertheless there are three operators which commute with the Hamiltonian and generate the $SL(2, R)$ algebra. By different Laughlin states we mean that we will find the Laughlin wavefunctions which are simultaneous eigenfunctions of Hamiltonian and different symmetry operators. In all cases we will show that the Laughlin states form a representation of $su_q(2)$.

We will also discuss the filling factor. The calculation of the filling factor (which is defined as the ratio of the total number of the electrons to the degeneracy of the first Landau level) is not clear in the non-compact surface, because the degeneracy and also the total area are both infinite in this case; therefore, we must calculate it in a different way. As will be seen, we will compute ν by using the plasma analogy.

In section 2 we will write the Laughlin states in such a way that they will be the eigenstates of the operators $\mathcal{L}_1^{-1}\mathcal{L}_2$ which were used in [12]. In section 3 another symmetry operator will be used (the operator $\mathcal{L}_2$ which generates dilation) and the single-particle and also the Laughlin wavefunctions will be found. By calculating the effective interaction potential, we will find the corresponding filling factor of these states. The generators of the quantum-group symmetry with B-dependent q will also be found. The degeneracy of the first Landau level which will be considered in sections 2 and 3 is infinite and the states are labelled by a continuous parameter. For completeness of our study, we will consider the discrete degenerate states in section 4.

2. Laughlin states as eigenstates of $\mathcal{L}_1^{-1}\mathcal{L}_2$

Consider the upper half-plane $\{z = x + \mathrm{i}y, y > 0\}$ with the metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}. \tag{1}$$

For a covariently constant magnetic field B and in the symmetric gauge $A_z = A_{\bar{z}} = B/2y$, the one-particle Hamiltonian is [11, 13]

$$H = -y^2\partial\bar{\partial} + \frac{\mathrm{i}B}{2}y(\partial + \bar{\partial}) + B^2/4. \tag{2}$$

(We take the electron mass $m = 2$.) The symmetry operators of this Hamiltonian are

$$\begin{aligned} L_1 &= \partial_x = \partial + \bar{\partial} \\ L_2 &= x\partial_x + y\partial_y = z\partial + \bar{z}\bar{\partial} \\ L_3 &= (y^2 - x^2)\partial_x - 2xy\partial_y - 2\mathrm{i}By. \end{aligned} \tag{3}$$

The operator L_i generates the $SL(2, R)$ algebra. The ground states with energy $B/4$ are [13]

$$\psi_0(z, \bar{z}) = y^B f(z) \tag{4}$$

where $f(z)$ is an arbitrary holomorphic function. In [13] it was shown that if we demand that $\psi_0(z, \bar{z})$ be an eigenfunction of $b = L_1^{-1}L_2$ with eigenvalue λ†, it takes the form

$$\psi_0(\lambda|z, \bar{z}) = y^B(\lambda - z)^{-B}. \tag{5}$$

If we define $T_\xi = \exp(\xi_1 c + \xi_2 b)$, where $c = L_1$, then it was shown that the operators

$$J_+ = \frac{T_\xi - T_\eta}{q - q^{-1}} \qquad J_- = \frac{T_{-\xi} - T_{-\eta}}{q - q^{-1}} \qquad q^{2J_0} = T_{\xi-\eta} \tag{6}$$

† By solving the eigenvalue problem $L_1^{-1}L_2\psi = \lambda\psi$ we mean solving the equation $(L_2 - \lambda L_1)\psi = 0$.

satisfy the $su_q(2)$ algebra [14]

$$[J_0, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = \frac{1}{q - q^{-1}}(q^{2J_0} - q^{-2J_0}) \tag{7}$$

and $\psi_0(\lambda|z, \bar{z})$ is a representation of this algebra. In equation (6) $\boldsymbol{\xi} = (\xi_1, \xi_2)$ and $\boldsymbol{\eta} = (\xi_1, -\xi_2)$.

Now to construct the N-particle wavefunction we assume the magnetic field to be so strong that we can approximately neglect the electron–electron interactions. In this case the Laughlin wavefunction takes the form

$$\psi_m(z_i, \bar{z}_i) = \prod_{j<k}^{N} (z_j - z_k)^m f(z_1, \ldots, \bar{z}_N). \tag{8}$$

We will take $f(z_i, \bar{z}_i)$ to be totally symmetric under the interchange $z_i \leftrightarrow z_j$ so that, with m an odd positive integer, ψ_m will be totally antisymmetric. $f(z_i, \bar{z}_i)$ must be found such that ψ_m will be the ground-state wavefunction of the non-interacting Hamiltonian $H = \sum_{i=1}^{N} H_i$, where H_i is defined as in equation (2), with energy $NB/4$. In this way it can be seen that $f(z_i, \bar{z}_i)$ is

$$f(z_1, \ldots, \bar{z}_N) = \prod_{i=1}^{N} y_i^B \psi(z_1, \ldots, z_N) \mathrm{e}^{\lambda_1 \bar{z}_1 + \cdots + \lambda_N \bar{z}_n} \tag{9}$$

with $\sum_{i=1}^{N} \lambda_i = 0$. The condition of symmetrization of $f(z_i, \bar{z}_i)$, forces us to take all λ_i as equal and therefore $\lambda_i = 0$, and $\psi(z_1, \ldots, z_N)$ as equal to $\prod_{i=1}^{N} \psi_0(z_i)$. So

$$\psi_m(z_i, \bar{z}_i) = \prod_{j<k}^{N} (z_j - z_k)^m \prod_{i=1}^{N} y_i^B \psi_0(z_i). \tag{10}$$

Now we will determine $\psi_0(z_i)$ such that ψ_m will be an eigenfunction of $\mathcal{L}_1^{-1}\mathcal{L}_2$ with eigenvalue λ. $\mathcal{L}_1$ and $\mathcal{L}_2$ are

$$\mathcal{L}_1 = \sum_{i=1}^{N} (\partial_i + \bar{\partial}_i) \qquad \mathcal{L}_2 = \sum_{i=1}^{N} (z_i \partial_i + \bar{z}_i \bar{\partial}_i). \tag{11}$$

By using the following relations

$$\begin{aligned} &\mathcal{L}_1 \prod_{j<k}^{N} (z_j - z_k)^m = 0 \\ &\mathcal{L}_2 \prod_{j<k}^{N} (z_j - z_k)^m = \frac{mN(N-1)}{2} \prod_{j<k}^{N} (z_j - z_k)^m \end{aligned} \tag{12}$$

and by using the condition of symmetrization of $\psi(z_1, \ldots, z_N)$, we obtain

$$\psi_m(\lambda, z_i, \bar{z}_i) = \prod_{j<k}^{N} (z_j - z_k)^m \prod_{i=1}^{N} y_i^B (\lambda - z_i)^{-B - m(N-1)/2}. \tag{13}$$

It can be seen that for $N = 1$, ψ_m reduces to equation (5). Also it can be checked that the above states form an infinite-dimensional representation of $su_q(2)$ algebra

$$\begin{aligned} &J_\pm \psi_m(\lambda, z_i, \bar{z}_i) = [1/2 \mp \lambda/\xi_1]_q \psi_m(\lambda \mp \xi_1, z_i, \bar{z}_i) \\ &q^{\pm J_0} \psi_m(\lambda, z_i, \bar{z}_i) = q^{\mp \lambda/\xi_1} \psi_m(\lambda, z_i, \bar{z}_i) \end{aligned} \tag{14}$$

where $[x]_q$ is defined by $[x]_q = (q^x - q^{-x})/(q - q^{-1})$ and $J_\pm$ and J_0 are defined in the same way as equation (6), with $T_{\bar{\xi}} = \exp(\xi_1 \mathcal{L}_1 + \xi_2 \mathcal{L}_1^{-1} \mathcal{L}_2)$.

For better understanding of the physics behind the Laughlin states, we will find another representation of the Laughlin states which are more suitable.

3. Laughlin wavefunction as eigenstates of $\mathcal{L}_2$

Let us first consider the one-particle wavefunction. If we demand that the state (4) be an eigenstate of the operator L_2 with eigenvalue m, it can easily be found that

$$|m\rangle = \psi_m(z, \bar{z}) = y^B z^{m-B}. \tag{15}$$

These states form a representation of the quantum group $su_q(2)$. This can be seen as follows. Define $E^\pm$ and k as

$$E^+ = -z[L_2 + \alpha + \beta]_q \qquad E^- = z^{-1}[L_2 + \alpha - \beta]_q \qquad k = q^{L_2+\alpha}. \tag{16}$$

Then we can verify that

$$[E^+, E^-]|m\rangle = \frac{k^2 - k^{-2}}{q - q^{-1}}|m\rangle \qquad kE^\pm k^{-1}|m\rangle = q^\pm E^\pm|m\rangle. \tag{17}$$

For the N-particle state we can see that, under the same assumptions as in the last section, the following wavefunction

$$\psi_m(z_i, \bar{z}_i) = \prod_{i=1}^{N} y_i^B \prod_{i<j}^{N} (z_i - z_j)^{m-B} \tag{18}$$

is

(i) an eigenstate of $H = \sum H_i$ with eigenvalue $NB/4$;

(ii) an eigenstate of

$$\mathcal{L} = \frac{2}{N(N-1)}\left(\mathcal{L}_2 + \frac{NB(N-3)}{2}\right)$$

with eigenvalue m,

(iii) totally antisymmetric.

The generators of $su_q(2)$ are now

$$E^+ = -\prod_{i<j}^{N}(z_i - z_j)[\mathcal{L} + \alpha + \beta]_q \qquad E^- = \prod_{i<j}^{N}(z_i - z_j)^{-1}[\mathcal{L} + \alpha - \beta]_q \tag{19}$$

$$k = q^{\mathcal{L}+\alpha}.$$

To ensure the Fermi–Dirac statistics for the wavefunction (18), $m - B$ must be 1, 3, 5, …. Since E^- is a lowering operator, and reduces m by 1, there should exist a lowest state $|m_{\min}\rangle$

$$E^-|m_{\min}\rangle = 0. \tag{20}$$

This condition can determine the deformation parameter q as (by choosing $a = \beta$)

$$q = \exp\left(\frac{\pi \mathrm{i}}{B+1}\right). \tag{21}$$

This equation relates q to the magnetic field as in the cases of the plane [7], sphere [9] and torus [5].

To calculate the filling factor that corresponds to the Laughlin state (18), we proceed to the same method that was followed by Laughlin [3], that is we introduce the effective classical potential energy ϕ in $|\psi_m|^2 = \mathrm{e}^{-\beta\phi}$. If we set the arbitrary effective temperature $1/\beta$ equal to $m - B$, we find

$$\phi = -(m-B)^2 \sum_{i<j}^{N} \ln|z_i - z_j|^2 - (m-B)\sum_i \ln y_i^{2B}. \tag{22}$$

The first term is the natural coulomb interaction of the particles with charge $m - B$. This is because the solution of the Laplace equation in the Poincaré half-plane is logarithmic. If one calculates the Laplace–Beltrami operator for the metric (1), one finds

$$\nabla^2\phi = \frac{1}{\sqrt{g}}\partial_i\sqrt{g}g^{ij}\partial_j\phi = y^2(\partial_x^2 + \partial_y^2)\phi. \tag{23}$$

Then

$$\nabla^2 \ln z\bar{z} = \Delta(\boldsymbol{r}) = y^2\delta(x)\delta(y) \tag{24}$$

where $\Delta(\boldsymbol{r})$ is the delta function on the Poincaré half-plane

$$\int \Delta(\boldsymbol{r})\sqrt{g}\,\mathrm{d}x\,\mathrm{d}y = 1. \tag{25}$$

The second term of equation (22) is the interaction of these particles with the uniform neutralizing background of charge density $\rho_0 = -B/2\pi$

$$\nabla^2(-\ln y^{2B}) = -4\pi\left(\frac{-B}{2\pi}\right). \tag{26}$$

But the plasma must be electrically neutral everywhere, so the total charge of these particles must be equal to the background charge, and this leads to the charge density

$$\rho_m = \frac{N}{A} = \frac{\rho_0}{m-B} \tag{27}$$

where N is the total number of the charged particles and A is total area. Therefore the filling factor $\nu = \rho_m/\rho_0$ is equal to

$$\nu = \frac{1}{m-B}. \tag{28}$$

So the wavefunction (18) corresponds to the filling factor $\nu = 1/M$ where $M = m - B$ is a positive odd integer.

4. Discrete representation

To complete our study of the FQHE on the Poincaré half-plane, we are going to discuss the Laughlin state as a discrete representation of the $su(1,1)$ algebra. As discussed in [11], if we define

$$J_0 = -\tfrac{1}{2}\mathrm{i}(L_1 - L_3) \qquad J_1 = -\tfrac{1}{2}\mathrm{i}(L_1 + L_3) \qquad J_2 = -\mathrm{i}L_2 \tag{29}$$

then it can be seen that they satisfy the $su(1,1)$ algebra:

$$[J_0, J_1] = \mathrm{i}J_2 \qquad [J_0, J_2] = -\mathrm{i}J_1 \qquad [J_1, J_2] = -\mathrm{i}J0 \tag{30}$$

and the Hamiltonian (2) becomes the Casimir, $C = J_0^2 - J_1^2 - J_2^2 = -4H + B^2$. This algebra has two kinds of representation, the discrete and continuous. These representations are labelled by the eigenvalues of the Casimir operator and the compact operator J_0

$$C|j,n\rangle = j(j+1)|j,n\rangle \qquad J_0|j,n\rangle = n|j,n\rangle \qquad \langle jn'|jn\rangle = \delta_{nn'} \tag{31}$$

The unitary irreducible representation of the discrete series is divided into two kinds D_j^+ or D_j^-, depending on the values of j, for $j > 0$

$$D_j^+ = \{|j, j+1\rangle, |j, j+2\rangle, \ldots\} \tag{32}$$

with $J_-|j, j+1\rangle = 0$, and for $j < 0$

$$D_j^- = \{|j, j\rangle, |j, j-1\rangle, \ldots\} \tag{33}$$

with $J_+|j, j\rangle = 0$. $J_\pm$ are as usual $J_1 \pm \mathrm{i}J_2$.

Now if we choose the eigenstates of the Hamiltonian to be in the discrete series, then j takes the values $-B+n$ where $n = 0, 1, 2, \ldots$. The ground states correspond to $j = -B$ and, therefore, we are in the D_j^- series. We have infinite discrete degenerate ground states

$$|-B, -B\rangle, |-B, -B-1\rangle, \ldots. \tag{34}$$

To find these states explicitly, we choose the ground state (4) to be the eigenstates of J_0 with eigenvalue n. By some calculation, we find

$$\psi_n(z, \bar{z}) = y^B \frac{(z-\mathrm{i})^{n-B}}{(z+\mathrm{i})^{n+B}}. \tag{35}$$

The quantum-group generators are the same as those in equation (16), by replacing z in equation (16) with $(z-\mathrm{i})/(z+\mathrm{i})$ and L_2 with J_0. Also, as our states are those in equation (34), so $n_{\max} = -B$ and therefore $E^+|n_{\max}\rangle = 0$ which gives q (by choosing $\alpha = \beta = -\frac{1}{2}$) as

$$q = \exp\left(\frac{\pi \mathrm{i}}{B+1}\right). \tag{36}$$

By the same reasoning, the N-particle wavefunction is $\psi_m(z_i, \bar{z}_i)$ in equation (10), where we determine $\psi_0(z_i)$ such that the $\psi_m(z_i, \bar{z}_i)$ will be the eigenfunction of $J = \sum_i J_0^i$ with eigenvalue M. A lengthy calculation shows that

$$\psi_m^M(z_i, \bar{z}_i) = \prod_{j<k}^{N}(z_j - z_k)^m \prod_{j=1}^{N} y_j^B \frac{(z_j-\mathrm{i})^{M/N-B-m(N-1)/2}}{(z_j+\mathrm{i})^{M/N+B+m(N-1)/2}} \tag{37}$$

with $m = 2k+1$. Finally the suitable $su_q(2)$ generators are

$$E^+ = -\prod_{j=1}^{N}\left(\frac{z_j-\mathrm{i}}{z_j+\mathrm{i}}\right)^{1/N}[J+\alpha+\beta]_q \qquad E^- = \prod_{j=1}^{N}\left(\frac{z_j-\mathrm{i}}{z_j+\mathrm{i}}\right)^{-1/N}[J+\alpha-\beta]_q \tag{38}$$

$k = q^{J+\alpha}$.

5. Conclusion

As mentioned in the introduction, one way to describe the behaviour of the electron in the FQHE is the concept of incompressible fluid, and its presence can be seen by checking the existance of the quantum group symmetry of the Laughlin states. In this paper we showed that in all cases there are such symmetries† and therefore we believe that this indicates that the collective motion of the electrons in FQHE on the Poincaré half-plane are also incompressible.

The last point is that it can be easily shown that the operator

$$L = g_1(z)L_1 + g_2(z)L_2 + g_3(z)L_3 + g_4(z) \tag{39}$$

† The generators of the $su_q(2)$ algebra in section 2 commute with the Hamiltonian, and these generators in sections 3 and 4 commute with H at the level of the ground states (see equation (40)).

with arbitrary holomorphic functions $g_i(z)$s, commutes with the Hamiltonian at the level of the ground state

$$[H, L]\psi_0(z, \bar{z}) = 0. \tag{40}$$

It can be shown that one can write the N-particle wavefunction to be the eigenstate of L, and with suitable choosing of $g_i(z)$, these functions can be made normal. The importance of this point will appear when we consider that the wavefunctions of the previous sections are not normal.

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